A note on sufficient conditions for positivity in \( x \in [-1, 1] \)

E. I. JURY† and M. MANSOUR‡

In this note, sufficient conditions for a real polynomial to be positive in the interval \([-1, 1]\) are presented. One such condition is obtained by using a nonlinear transformation which maps the interval \([-1, 1]\) onto the periphery of the unit circle. The root distribution of the transformed polynomial determines the positivity condition.

1. Introduction

In recent years several applications arise in which one has to test a real polynomial for positivity in the interval \([-1, 1]\). Among such applications are stability testing of delay differential systems (Jury and Mansour 1982) and stability of multidimensional discrete systems (Bose 1982). In Jury and Mansour (1982) explicit conditions for positivity up to a quartic equation are presented. For higher-order polynomials it is often desirable to obtain some sufficient conditions. One such condition, which is a weak one, was also given by Jury and Mansour (1982). In this note, stronger sufficient conditions are given. Also, in this note a correction presented by Kamen (1983) for testing stability of delay systems independent of delay, is clarified in connection with our earlier contribution (Jury and Mansour 1982). By using a non-linear transformation introduced recently by Lai (1984), we can derive the required sufficient condition. An example is given to indicate the testing procedure.

2. Formulation of the sufficiency condition

Let

\[
F(z) = \sum_{n=0}^{N} a_n z^n, \quad a_n > 0 \text{ and } N \text{ even, and } F(0) > 0
\]

Steps

(i) Check if \( F(z) \) is stable, i.e.

\[
F(z) \neq 0, \quad |z| \geq 1
\]

(ii) If step (i) is satisfied, then let

\[
z \rightarrow \frac{1}{2} \left[ \frac{w^2 + 1}{w} \right]
\]

and form

\[
F_{2N}(w) = w^N[F_N^*(w)] + F_N(w)
\]

Received 25 March 1985
† Department of Electrical and Computer Engineering, University of Miami, Coral Gables, Florida 33124, U.S.A.
‡ Institute for Automatic Control and Industrial Electronics, Swiss Federal Institute of Technology, CH-8092 Zürich, Switzerland.
where

\[ F_N(w) = w^n \left[ F_N \left( \frac{1}{w} \right) \right] \quad (5) \]

The transformation (3) maps all real roots of \( F(z) \) inside the unit circle onto the periphery of the unit circle in the \( w \)-plane, and any complex pair of root becomes two complex pairs symmetrical with respect to the unit circle in the \( w \)-plane, one pair inside and the other outside the unit circle (Lai 1984). Therefore the condition for positivity, i.e.

\[ F(z) > 0 \quad (z \in [-1, 1]) \quad (6) \]

reduces to requiring all the roots in the \( w \)-plane to be symmetric with respect to the unit circle, and none on the unit circle. Such a condition can be ascertained as follows.

(iii) Form

\[ \left[ \frac{dF_{2N}(w)}{dw} \right]^* = [F_{2N}(w)]^* \quad (7) \]

where the star represents the reciprocal polynomial.

Following Cohn (1914) and Jury (1964, 1982), (7) requires that \( N \) of its roots should lie inside the unit circle. Since \( F_{2N}(w) \) is a reciprocal polynomial, \( N \) of its complex roots are inside the unit circle and \( N \) are outside. Hence there exist no roots on the unit circle, and thus no real root of \( F(z) \) exists in the interval \([-1, 1]\), i.e. a sufficient condition for positivity.

Another obvious sufficient condition for positivity can be obtained by requiring all the roots of \( F(z) \) for any \( N \) to be outside the unit circle. Such a condition can easily be obtained (Jury 1964, 1982).

3. Example

Let

\[ F(z) = z^2 - 0.5z + 0.5 \quad (8) \]

The above is a positive polynomial, for it has two complex roots inside the unit circle and thus no root in the interval \([-1, 1]\).

Steps

(i) Let

\[ z \rightarrow \frac{1}{2} \left[ \frac{w^2 + 1}{w} \right] \quad (9) \]

in \( F(z) \) and form \( F_4(w) \):

\[ F_4(w) = w^2[w^2 - w + 2] + 2w^2 - w + 1. \quad (10) \]

or

\[ F_4(w) = w^4 - w^3 + 4w^2 - w + 1 \quad (11) \]
Sufficient conditions for positivity in \([-1, 1]\)

(ii) Form

\[
\left[ F_4(w) \right]^* = [4w^3 - 3w^2 + 8w - 1]^* = 4 - 3w + 8w^2 - w^3
\]

(12)

The above polynomial has two roots inside the unit circle, and hence \(F_4(w)\) has two complex roots inside and two complex roots outside the unit circle, and no roots in the unit circle. Therefore \(F(z)\) has no real roots and thus is positive.

Remark

Following Kamen's (1983) corrections, equations (4) and (5) of Jury and Mansour (1982) are not equivalent. They are so, if we add the following to (4):

\[ P(0, z) \neq 0, \quad |z| = |\exp(-hs)| = 1 \quad (4a) \]

4. Conclusions

In this note two sufficient conditions for positivity of a real polynomial in the interval \([-1, 1]\) have been obtained. One such condition is a strong one, and was obtained by using a non-linear transformation which maps the real axis \([-1, 1]\) onto the periphery of the unit circle. The condition of non-existence of real roots in \([-1, 1]\) can be ascertained from a certain root distribution of the transformed polynomial.

Acknowledgments

Research of E. I. Jury was supported by NSF Grant ECS-8410298.

References


